LINEAR ALGEBRA

B.Sc II year

Topics

- Characteristic Vector
- Characteristic Value
- Characteristic Polynomial
- Characteristic Equation
- Cayley Hamilton Theorem

Characteristic Vector, Characteristic Value of a matrix

- Any non-zero vector X is said to be a characteristic vector of a square matrix A if there exists a scalar such that $AX = \lambda X$.
- Here A can be a mxn matrix and X can be a n * 1 matrix.
- Then λ is said to be a characteristic value of the matrix A corresponding to the characteristic vector X .
- Also X is said to be characteristic vector corresponding to the characteristic value λ of the matrix A.
- Characteristic vectors are sometimes called Proper or Latent or Eigen vectors.

Characteristic Polynomial

- Let $A=[a_{ij}]$ n×n λ and any indeterminate scalar. The matrix $A - \lambda I$ is called the characteristic matrix of A, where I is the unit matrix of order n.
- Also

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix},$$

a polynomial in λ of degree n, is called the characteristic polynomial of A .

Characteristic Equation

- The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A and its roots (the values of λ) are called characteristic roots or eigenvalues.
- It is also known that every square matrix has its characteristic equation.

Cayley Hamilton Theorem

Statement :

Every square matrix satisfies its characteristic equation.

Proof :

Let A be a n- rowed square matrix ie.,

Let $A = [a_{ij}]_{n*n}$.

The characteristic equation of A is given by

$$|A - \lambda I| = f(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots\cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots\cdots + a_n]$$

where a_i's $\in F$.

Since all the elements of $(A-\lambda I)$ are at most of Ist degree in λ .

All the elements of adj $(A-\lambda I)$ are polynomials in λ of degree (n-1) or less and hence adj $(A-\lambda I)$ can be expressed a matrix polynozmial in λ.

Let adj $(A-\lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \ldots\ldots + B_{n-2} \lambda^1 + B_{n-1}$

where $B_0, B_1, \ldots\ldots B_{n-1}$ are n- rowed square matrices.

Now $(A-\lambda I)$ adj $(A-\lambda I) = |A - \lambda I| I$

$[\because A(\text{adj}A) = = |A| I]$

$$\Rightarrow (A-\lambda)(B_0\lambda^{n-1}+B_1\lambda^{n-2}+\dots+B_{n-2}\lambda^1+B_{n-1})$$

$$= (-1)^n[\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\dots\dots+a_n]I$$

Comparing coefficients of like powers of λ, we obtain

$$-B_0 = (-1)^n I,$$

$$AB_0-B_1 = (-1)^n a_1 I,$$

$$AB_1-B_2 = (-1)^n a_2 I,$$

....

....

$$AB_{n-1} = (-1)^n a_n I.$$

Premultiplying the above equations successively by and adding $A^n, A^{n-1}, \ldots\ldots\ldots, I$ and adding,

we obtain

$$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \ldots. + (-1)^n a_n I$$

$$\Rightarrow (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots. + a_n I] = 0$$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots. + a_n I = 0$$

A satisfies its characteristic equation. (O is zero matrix)

$\therefore$ Every square matrix satisfies its characteristic equation

THE END