

# LINEAR ALGEBRA

B.Sc II year



# Topics

- Characteristic Vector
- Characteristic Value
- Characteristic Polynomial
- Characteristic Equation
- Cayley Hamilton Theorem

# Characteristic Vector,

## Characteristic Value of a matrix

- Any non-zero vector  $X$  is said to be a characteristic vector of a square matrix  $A$  if there exists a scalar such that  $AX = \lambda X$ .
- Here  $A$  can be a  $m \times n$  matrix and  $X$  can be a  $n \times 1$  matrix.
- Then  $\lambda$  is said to be a characteristic value of the matrix  $A$  corresponding to the characteristic vector  $X$ .
- Also  $X$  is said to be characteristic vector corresponding to the characteristic value  $\lambda$  of the matrix  $A$ .
- Characteristic vectors are sometimes called Proper or Latent or Eigen vectors.

# Characteristic Polynomial

- Let  $A=[a_{ij}]$   $n \times n$   $\lambda$  and any indeterminate scalar. The matrix  $A - \lambda I$  is called the characteristic matrix of  $A$ , where  $I$  is the unit matrix of order  $n$ .

- Also  $|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix},$

a polynomial in  $\lambda$  of degree  $n$ , is called the characteristic polynomial of  $A$ .



# Characteristic Equation

- The equation  $|A - \lambda I| = 0$  is called the characteristic equation of the matrix  $A$  and its roots (the values of  $\lambda$ ) are called characteristic roots or eigenvalues.
- It is also known that every square matrix has its characteristic equation.

# Cayley Hamilton Theorem

Statement :

Every square matrix satisfies its characteristic equation.

Proof :

Let A be a n- rowed square matrix ie.,

Let  $A = [a_{ij}]_{n \times n}$ .

The characteristic equation of A is given by

$$|A - \lambda I| = f(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n]$$

where  $a_i$ 's  $\in F$ .

Since all the elements of  $(A - \lambda I)$  are at most of 1st degree in  $\lambda$ .

All the elements of  $\text{adj}(A - \lambda I)$  are polynomials in  $\lambda$  of degree  $(n-1)$  or less and hence  $\text{adj}(A - \lambda I)$  can be expressed a matrix polynomial in  $\lambda$ .

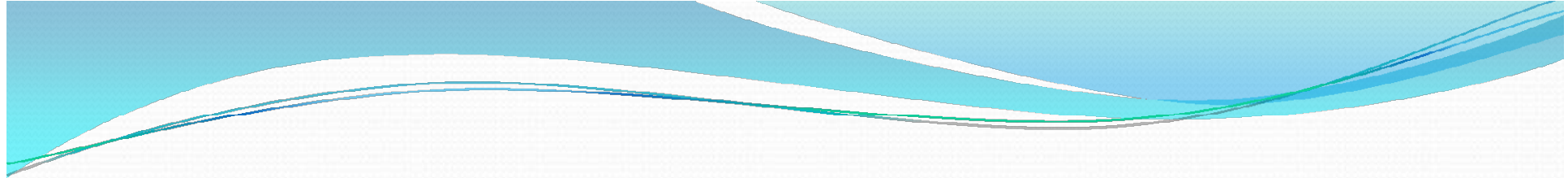
$$\text{Let } \text{adj}(A - \lambda I) = B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \dots + B_{n-2} \lambda^1 + B_{n-1}$$

where  $B_0, B_1, \dots, B_{n-1}$  are  $n$ -rowed square matrices.

$$\text{Now } (A - \lambda I) \text{adj}(A - \lambda I) = |A - \lambda I| I$$

$$[\because A(\text{adj}A) = |A| I]$$





$$\begin{aligned} \Rightarrow (A - \lambda) (B_0 \lambda^{n-1} + B_1 \lambda^{n-2} + \dots + B_{n-2} \lambda^1 + B_{n-1}) \\ = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n] I \end{aligned}$$

Comparing coefficients of like powers of  $\lambda$ , we obtain

$$\begin{aligned} -B_0 &= (-1)^n I, \\ AB_0 - B_1 &= (-1)^n a_1 I, \\ AB_1 - B_2 &= (-1)^n a_2 I, \\ &\dots\dots\dots \\ &\dots\dots\dots \\ AB_{n-1} &= (-1)^n a_n I. \end{aligned}$$

Premultiplying the above equations successively by  $A^{n-1}, A^{n-2}, \dots, I$  and adding,

we obtain

$$0 = (-1)^n A^n + (-1)^n a_1 A^{n-1} + (-1)^n a_2 A^{n-2} + \dots + (-1)^n a_n I$$

$$\Rightarrow (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I] = 0$$

$$\Rightarrow A^n + a_1 A^{n-1} + a_2 A^{n-2} + \dots + a_n I = 0$$

$A$  satisfies its characteristic equation. ( $O$  is zero matrix)

$\therefore$  Every square matrix satisfies its characteristic equation



THE END