GROUP THEORY

FIELDS

Field:

- A field is a commutative division ring.
- Another definition : Let F be a non empty set with atleast two elements and equipped with two binary operations defined by (+),(.) respectively. Then the algebraic structure F(+,.) is a field if the following properties are satisfied:
- Axioms of addition:

  1.Closure property:$a+b \in F$ for all $a,b \in F$.

2. Associative law:

$a+(b+c)=(a+b)+c$

3. Existence of identity: for all $a\in F$, there exists an element $0\in F$ Such that

$a+0=0+a$

4. Existence of inverse: for each $a\in F$, there exists an element $b\in F$ such that

$a+b=0=b+a$

Element b is called inverse of a and is denoted by $-a$.

Commutative law: a+b=b+a for all a,b$\in$ F

Axioms of multiplication:

6.Closure law : for all a,b$\in$F ,the elements of a.b$\in$F

7.Commutatibe law : a.b=b.a for all a,b$\in$F

8.Associative law: multiplication is associative

i.e.,a.(b.c)=(a.b).c for all a,b$\in$F

9.Existence of multiplication identity: for each a$\in$F , there exists 1$\in$F such that

a.1=a=1.a

10.Existence of inverse: for each non zero element $a \in F$, there exists an element $b \in F$ such that $a.b=1=b.a$

Element b is called multiplication inverse of a and is denoted by $1/a$

11.Distributive law : multiplication is distributive w.r.t. addition.

i.e.,for all $a,b,c \in F$

$$a.(b+c)=a.b+a.c$$

$$(b+c).a=b.a+c.a$$

Integral domain:

A commutative ring with unity having no zero divisor is called integral domain

Another definition:

Let D be a ring ,D is said to be Integral Domain if the following conditions are holds

- D is commutative (ab=ba)
- D has ring with unity (a.1=a=1.a)
- D has no zero divisor

Let R be a ring,an element a#0∈R is said to be zero divisor b# 0 ∈ R if ab=0.

Theorem:

Statement
Every field is an integral domain.
Proof:
Suppose (F,+,.) is a Field
i.E F is commutative ,ring with unity and Every non zero element is invertable.
Claim:
F is integral domain.
It is enough to prove that
F has no zero divisior

Let a & b be elements of F with $a \neq 0$
such that $ab = 0$.
Now, $a \neq 0$ implies that a^{-1} exists.
For $ab = 0$,
multiply a^{-1} to both sides,
$(ab)a^{-1} = (0)a^{-1}$
$(a.a^{-1})b = 0$ (1)
$b = 0$
$\Rightarrow \quad b = 0$
Therefore, $a \neq 0$,
$ab = 0$ implies that $b = 0$
Similarly,
let $ab = 0$ and $b \neq 0$
Let $a,b \in F$
Let a & b be elements of F with $a \neq 0$ such that
$ab = 0$.
Now, $a \neq 0$ implies that a-1 exists.

Now, $b \neq 0$ implies that b^{-1} exists.

For $ab = 0$,

multiply b^{-1} to both sides,

$(ab)b^{-1} = (0)b^{-1}$

$(b.b^{-1})a = 0$ (1)

$a = 0$

$\Rightarrow \quad a = 0$

Therefore, $b \neq 0$, $ab = 0$ implies that $a = 0$

In field F,

$ab = 0$

$\Rightarrow a = 0$ or $b = 0$

Therefore, F has no zero divisors.

Hence proved, Field is an integral domain.