MEAN VALUE THEOREM

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Subject : REAL ANLAYSIS

CURRICULUM

- **Mean Value Theorem (MVT)**
  - Lagrange's MVT
  - Rolle's Theorem
  - Cauchy's MVT
- **Applications**

MOTIVATION

- **Law of Mean:**

    For a "smooth" curve (a curve which can be drawn in a plane without lifting the pencil on a certain interval) y=f(x) (a≤x≤b) it looks evident that at some point c lies between a and b i.e. a<c<b, the slope of the tangent f'(c) will be equal to the slope of the chord joining the end points of the curve. That is, for some c lies between a and b (a<c<b),

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Physical Interpretation:

The velocity of a particle (matter) is exactly equal to the average speed.

MEAN VALUE THEOREM (LAGRANGE'S MVT)

- **Statement:**

Let f(x) be any real valued function defined in a≤x≤b such that

(i) f(x) is continuous in a≤x≤b

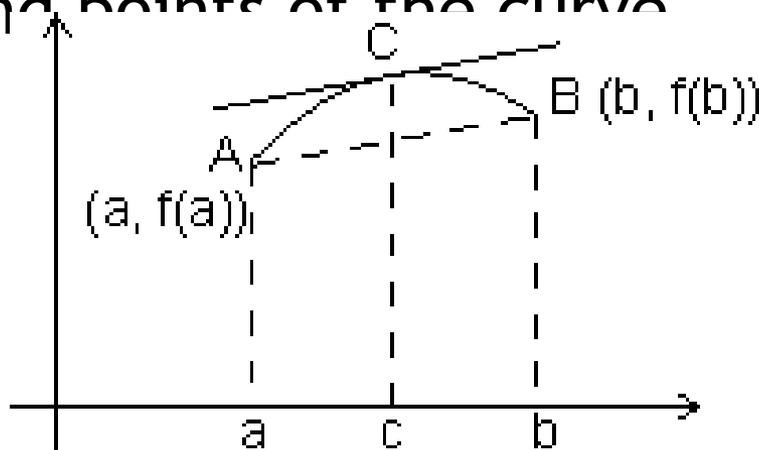(ii) f(x) is differentiable in a<x<b

Then, there exists at least one c in a<c<b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Alternative form (by taking b=a+h, h: small increment)

f(a+h)=f(a)+hf'(a+θh) for some θ, 0< θ<1

GEOMETRICAL INTERPRETATION OF MVT

- For a continuous curve y=f(x) defined in a≤x≤b, the slope of the tangent f'(c) (where c lies between a and b i.e. a<c<b) to the curve is parallel to the slope of the chord joining the end points of the curve.

SPECIAL CASE OF MVT

- **Rolle's Theorem**

Statement:

Let $f(x)$ be any real valued function defined in $a \leq x \leq b$ such that

(i) $f(x)$ is continuous in $a \leq x \leq b$

(ii) $f(x)$ is differentiable in $a < x < b$

(iii) $f(a) = f(b)$

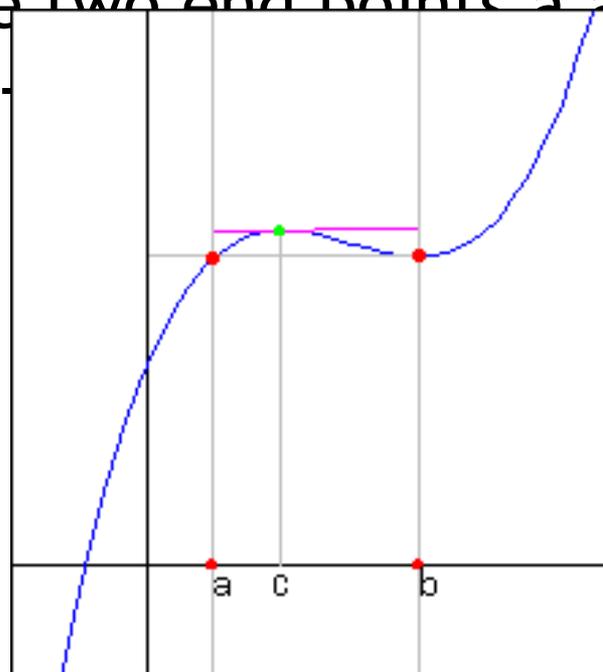Then, there exists at least one c in $a < c < b$ such that $f'(c) = 0$.

Note: All the conditions of Rolle's theorem are sufficient not necessary.

Counter Example: **i) $f(x) = 2 + (x-1)^{2/3}$ in $0 \leq x \leq 2$**

ii) $f(x) = |x-1| + |x-2|$ on $[-1,3]$

GEOMETRICAL INTERPRETATION OF ROLLE'S THEOREM

- For a continuous curve y=f(x) defined in a≤x≤b, the slope of the tangent f'(c) (where c lies between a and b i.e. a<c<b) to the curve joining the two end points a and b is parallel to the x-

GENERAL CASE (CAUCHY'S MVT)

- **Statement:**

Let f(x) and g(x) be two real valued function defined in a≤x≤b such that

(i) f(x) and g(x) are continuous in a≤x≤b

(ii) f(x) and g(x) are both differentiable in a<x<b

(iii) g'(x)≠0 for some a<x<b

Then, there exists at least one c in a<c<b such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

ALTERNATIVE FORM:

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, (0 < \theta < 1)$$

INTERPRETATION (CAUCHY'S MVT)

- Useful generalization of the law of mean by considering a smooth curve in parametric representation x=g(t) and y=f(t) (a≤t≤b).

- The slope of the tangent to the curve at t=c is $\dfrac{f'(c)}{g'(c)}$

- The generalized law of mean asserts that there is always a value of c in a<c<b, for which the slope of the curve is equal to the slope of the tangent at c.

OBSERVATIONS

Rolle's Theorem
By taking f(x)=x in MVT

MVT (Lagrange's)
By taking g(x)=x in Cauchy's MVT

Cauchy's MVT

APPLICATIONS

- To estimate some values of trignometrical function say $\sin 46^0$ etc.

- Darboux's theorem: If the interval is an open subset of R and f:I→R is differentiable at every point of I, then the range of an interval f' is an interval (not necessarily an open set).

  [This has the flavour of an "Internediate Value Theorem" for f', but we are not assuming f' to be continuous].

- L' Hospital's Rule: If f(x) →0, g(x) →0 and f'(x)/g'(x) →L as x→c, then f(x)/g(x)→L as x→c.

⊙ To deduce the necessary and sufficient condition of monotonic increasing or decreasing function.

For a continuous function f:[a,b]→R that is differentiable on (a,b), the following conditions are equivalent:

(i) f is increasing (or decreasing)

(ii) $f'(x) \geq 0$ (or $f'(x) \leq 0$)

➢ Not only the above examples but many more applications can found in different reference books from mathematics.

BOOKS RECOMMENDED

- A first course in real analysis: Sterling K. Berberian. Springer.

- Mathematical analysis: Tom M. Apostol. Pearson Education Inc.

- An introduction to analysis: Differential calculus. Ghosh and Maity. New Central Book Agency.

- Methods of Real analysis. Richard R Goldberg. Wiley.