LINEAR ALGEBRA

BSC IInd YEAR

INTERNAL COMPOSITION OR VECTOR ADDITION :

Let V be any non-empty set.

(+) be any operation (or addition) defined on the set V

If $\forall\ \alpha, \beta \in V \implies \alpha+\beta \in V$ then + is called an internal composition on the set V.

EXTERNAL COMPOSION OR SCALAR MULTIPLICATION :

Let V and F be two non-empty sets. ($\bullet$) be any operation defined on the set V.

Then $\forall$ a $\in$ F and $\alpha \in V \implies a \bullet \alpha \in V$ is called an external composition on the set V.

VECTOR SPACE :

DEFINITION :

Let V be a non-empty set whose elements are called vectors. Let F be any set whose elements are scalars where (F,+,.) is a Field

The set V is said to be a vector space if

1. There is defined an internal composition in V called addition of vectors denoted by +, for which

(V.+) is an abelian group.

2. There is defined an external composition in V over F, called the scalar multiplication in which

$a \in F$ and $\alpha \in V \Longrightarrow a \cdot \alpha \in V$

3.The above two compositions satisfy the fallowing postulates

i. $a(\alpha+\beta) = a\alpha+b\beta$

ii. $(a+b)\alpha = a\alpha+b\alpha$

iii. $(ab)\alpha = a(b\alpha)$

iv. $1.\alpha = \alpha$

THEOREM :

Let V(F) be a vector space. A non-empty set $W \subseteq V$. The necessary and sufficient condition for W to be a subspace of V is
$a, b \in F$ and $\alpha, \beta \in V \Longrightarrow a\alpha + b\beta \in W$

PROOF:

Let V(F) be a vector space and W V

NECESSARY CONDITION:

Suppose that W is a subspace of V.

Claim: W(F) is a subspace of V (F)

W (F) is a vector space

$\therefore$ $a \in F, \alpha \in W \Longrightarrow a\alpha \in W$ and $b \in F$, $\beta \in W \Longrightarrow b\beta \in W$

Now $a\alpha \in W, b\beta \in W \Longrightarrow a\alpha + b\beta \in W$

SUFFICIENT CONDITION:

Let W be the non-empty subset of V satisfying the given condition

i.e., a,b∈F and $\alpha, \beta \in W \Longrightarrow a\alpha + b\beta \in W \ldots (1)$

Taking a=1,b=-1 and $\alpha, \beta \in W \quad \Longrightarrow 1\alpha + (-1)\beta \in W$

$\Longrightarrow \quad \alpha\text{-}\beta \in W \qquad [\because \ \alpha \epsilon W \Longrightarrow \alpha \epsilon V \ and \ 1\alpha = \alpha \ in \ V]$

(H ⊆G and a,b$\epsilon H \Longrightarrow a o b^{-1} \epsilon$ H then (H, o) is subgroup of (G, 0))

⇒(W,+) is a subgroup of the abelian group (V,+)

⟹ (W,+) is an abelian group

Again taking b=0

a, $0 \in \mathrm{F}$ and $\alpha, \beta \in W \Rightarrow a\alpha + 0\beta \epsilon W \Rightarrow a\alpha \epsilon W \Longrightarrow a \in F \ and \ \alpha \in W \Longrightarrow a\alpha \in W$

∴ W is closed under scalar multiplication

The remaining postulates of vector space hold in W as W⊆ V

∴ W(F) is a vector subspace of V (F).

THEOREM:

A non-empty set W is a subset of vector space V(F). W is a subspace of W if only if a∈F and $\alpha, \beta \epsilon V \Longrightarrow a\alpha + \beta \in W$.

PROOF:

Let V(F) be a vector space and W$\subseteq V$

NECESSARY CONDITION:

Suppose that W is a subspace of V.

Claim: W(F) is a subspace of V (F)

W (F) is a vector space

$\therefore$ a ∈ F ,α ∈ W ⟹ aα∈ W

Further aα∈ W , $\beta \in W \Longrightarrow a\alpha + \beta \in W$

SUFFICIENT CONDITION:

Let W be the non-empty subset of V satisfying the given condition

i.e., a∈F and α,$\beta \in W \implies a\alpha + \beta \in W$

(1) Now taking a=-1,for $\alpha \epsilon W$

we have, (-1)$\alpha + \alpha \epsilon W \implies \bar{O} \in W$

(2) Again a∈F,α,$\bar{O} \in W \implies a\alpha + \bar{O} \in W \implies$a$\alpha \epsilon W$

∴ W is closed under scalar multiplication

(3) -1∈ F and α,$\bar{O} \in W \implies$ (-1)α+ $\bar{O} \in W \implies$ -α ∈ W

∴Inverse exists in W

The remaining postulates of vector space hold in W as W⊆ V

∴ W(F) is a vector subspace of V (F).

THEROEM:

Let V(F) be a vector space. A non-empty set W V. The necessary and sufficient condition for W to be a subspace of V are

(1)$\alpha \epsilon W, \beta \epsilon W \implies \alpha - \beta \epsilon W$

(2) a ∈ F ,α ∈ W ⟹ aα∈ W

PROOF:

Let V(F) be a vector space

NECESSARY CONDITION:

(1) W is a vector subspace of V

⟹W is a subgroup of (V,+) ⟹ (W, +) is a group

⟹ if α,β ∈ W then α-β ∈ W

(2) W is a subspace of V

⇒ W is closed under scalar multiplication ⟹ for a ∈ F, α ∈W ; aα ∈W

SUFFICIENT CONDITION:

Let W be a nonempty subset of V satisfying the two given conditions

$\alpha \in W, \alpha \in W \Longrightarrow \alpha - \alpha \in W \Longrightarrow \bar{O} \in W$

$\therefore$The zero vector of V is also the zero vector of W

$\bar{O} \in W, \alpha \in W \Longrightarrow \bar{O} - \alpha \in W \Longrightarrow (-\alpha \in W)$

$\Longrightarrow$ additive inverse of each element of W is also in W

Again $\alpha \in W, \beta \in W \Longrightarrow \alpha \in W, (-\beta) \in W \Longrightarrow \alpha - (-\beta) \in W$

$$\Longrightarrow \alpha + \beta \in W$$

i.e., W is a closed under vector addition

W $\subseteq$V , all the elements of W are also the elements of V.

Thereby vector addition in W will be associative and commutative.

This implies that (W ,+) is an abelian group.

Further by (2), W is closed under scalar multiplication and the other postulates of vector space hold in w as W $\subseteq$ V

$\therefore$ W itself is a vector space under the operations of V.

Hence W (F) is a vector subspace of V(F)

THANK YOU